The ANTI-N-QUEENS Problem

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The N-QUEENS problem is a well-known problem and it consists of placing n non-attacking queens on an $n \times n$ chessboard. It is solvable for any $n > 3$.
In this paper, we will explore a related similar problem, tentatively named ANTI-N-QUEENS. As guessed from the name, the goal of this problem is different. A simple formulatin of the problem is the following:

> What is the maximum number of "safe" squares when N queens are placed on a board?

We defined a square on the board as *safe* when it is not threatened by any queen on the board. There is no restriction that the queens are not attacking each other, nor (obviously) that any rule of chess must be followed.
In this paper, in each case where a solution is presented, a single example board is shown, out of all possible ones. This is the one with the queens most close to the top-left corner. More formally, it's the first board when all possible solutions have been lexicographically sorted by row (starting from the top) and then by column (starting from the left).

Observations

Observation 1. While the N-QUEENS problem is a canonical example of solving via backtracking, going from a brute-force $O(n^n)$ to a polynomial one, the ANTI-N-QUEENS problem cannot be solved so fast.

Observation 2. All possible arrangements of N queens on a square board of side size M are obviously all the combinations of N elements out of a set of M, or $\binom{M}{N}$, where:

$$\binom{M}{N} = \frac{M!}{N!(M-N)!}$$

Observation 3. Another observation stems from the fact that, if a queen placed on a square S_Q does not threaten a square S_f, then the reverse is also true: a queen placed on S_f can not threaten the square S_Q. Therefore, if, for a given board, we have a configuration with N queens and F safe squares, we can generate the reverse configuration with F queens and N safe squares, which will be valid (albeit maybe not optimal).

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Boards of variant size

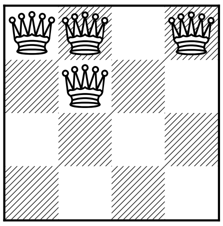
We first explore the problem with N queens on a $N \times N$ board. We will present the *final* results for board sizes up to 16×16.
The possible combinations are $\binom{N^2}{N}$.

Sizes of 1, 2, and 3

It should be obvious that for small boards where $N \leq 3$, there can be no placement of N queens that allows any square to be safe.
In the 1×1 case, the single cell is occupied by the queen. In the 2×2 case, all cells are in the same row or column with at least one queen. Finally, in the 3×3 case, all cells are in the same row, column, or diagonal with at least one queen.

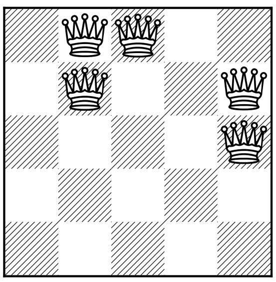
Size 4

On a board of size 4×4 with 4 queens, the maximum number of safe squares is 1. Out of 1 820 possible combinations, this is achieved in 184.



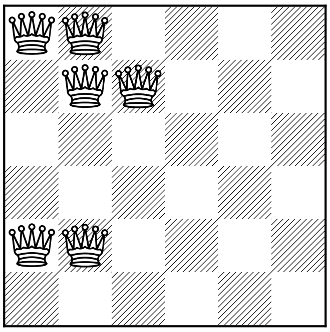
Size 5

On a board of size 5×5 with 5 queens, the maximum number of safe squares is 3. Out of 53 130 possible combinations, this is achieved in 8.



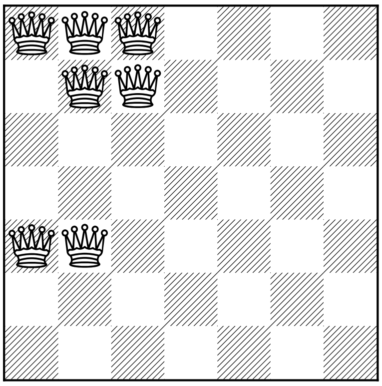
Size 6

On a board of size 6×6 with 6 queens, the maximum number of safe squares is 5. Out of 1 947 792 possible combinations, this is achieved in 24.



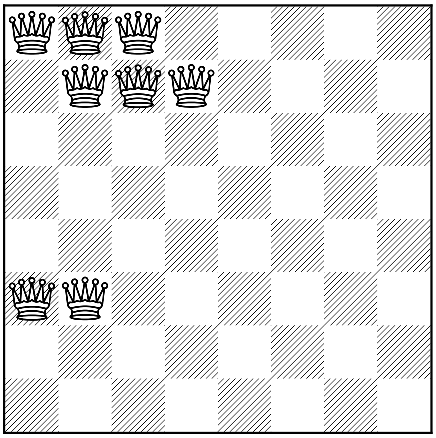
Size 7

On a board of size 7×7 with 7 queens, the maximum number of safe squares is 7. Out of 85 900 584 possible combinations, this is achieved in 304.



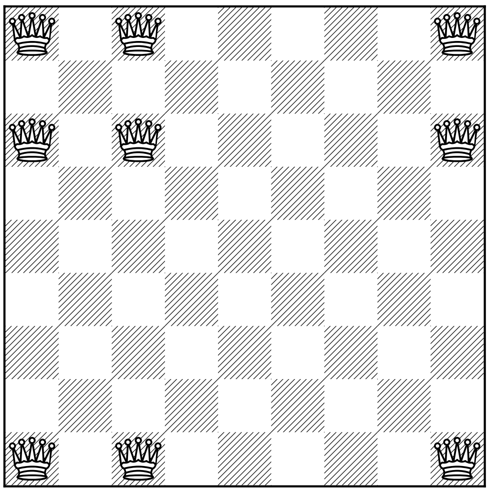
Size 8

On a board of size 8×8 with 8 queens, the maximum number of safe squares is 11. Out of 131 198 072 possible combinations, this is achieved in 48.



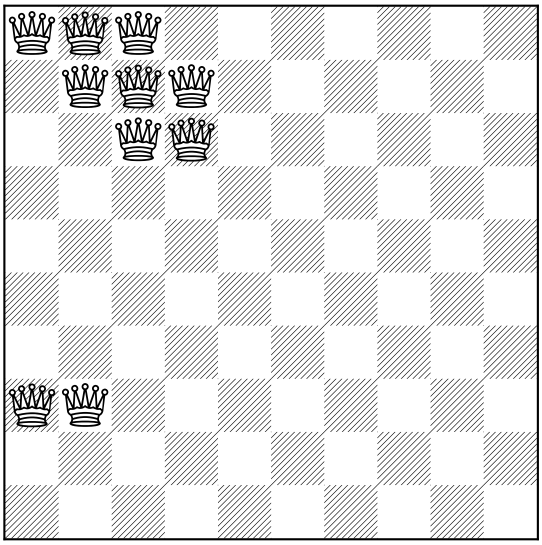
Size 9

On a board of size 9×9 with 9 queens, the maximum number of safe squares is 18. Out of 260 887 834 350 possible combinations, this is achieved in 4.



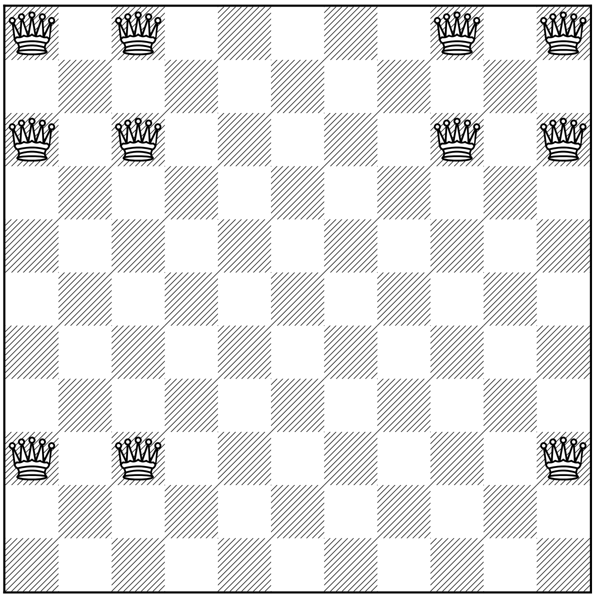
Size 10

On a board of size 10×10 with 10 queens, the maximum number of safe squares is 22. Out of 17 310 309 456 440 possible combinations, this is achieved in 8.



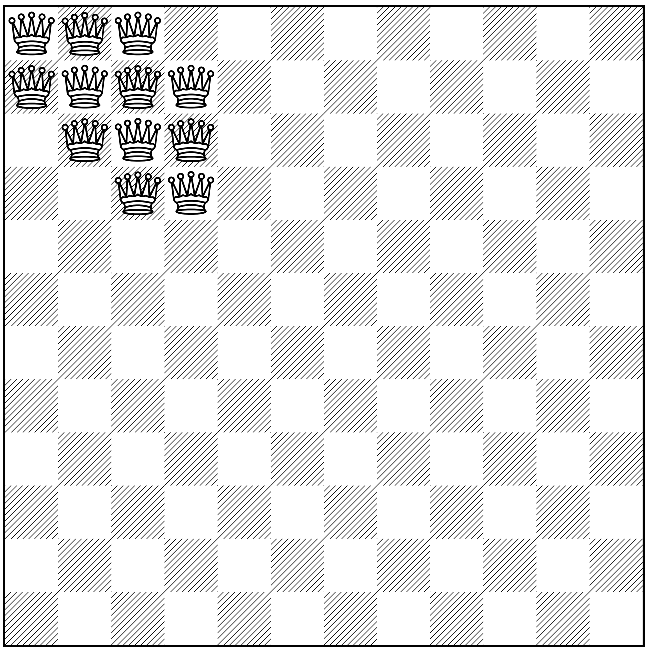
Size 11

On a board of size 11×11 with 11 queens, the maximum number of safe squares is 30. Out of 1 276 749 965 026 536 possible combinations, this is achieved in 16.



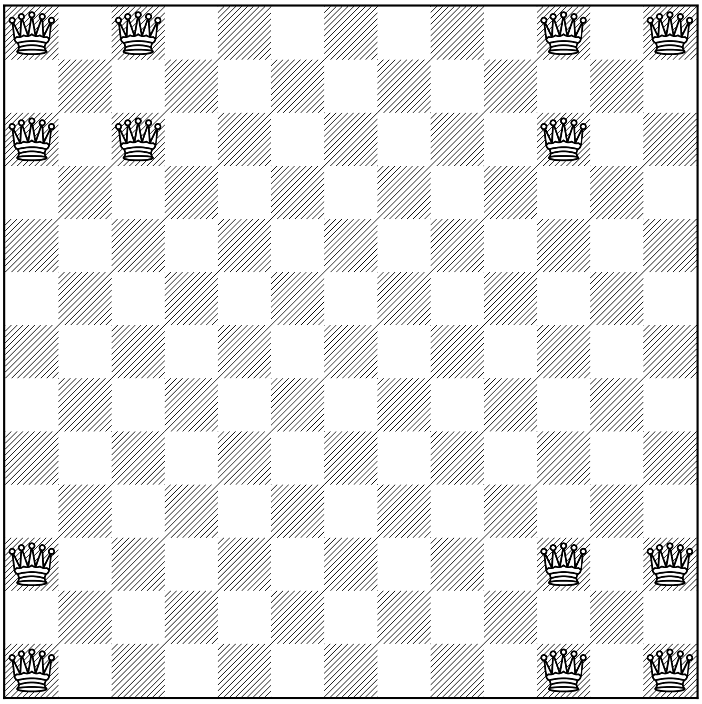
Size 12

On a board of size 12×12 with 12 queens, the maximum number of safe squares is 36. Out of 103 619 293 824 707 388 possible combinations, this is achieved in 42.



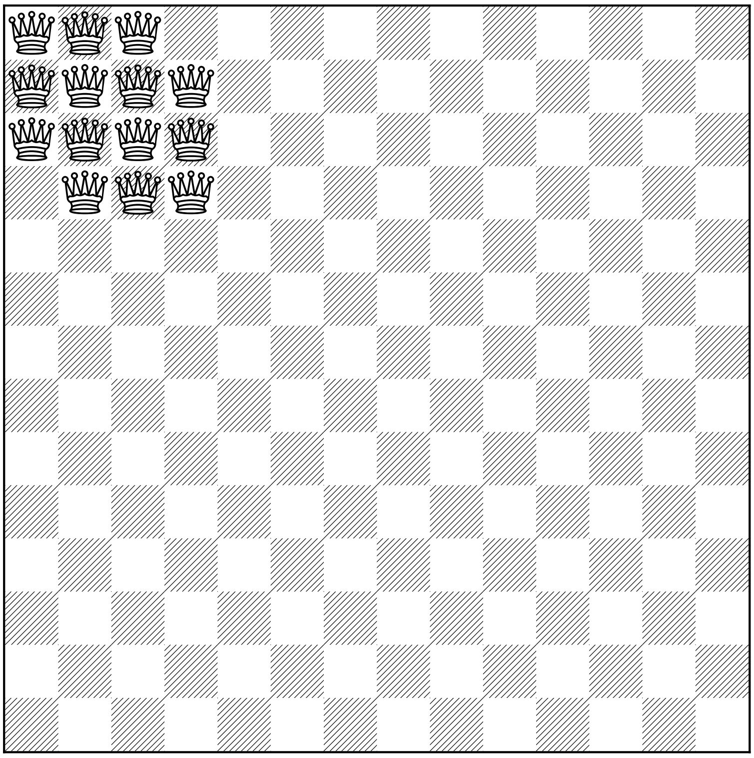
Size 13

On a board of size 13×13 with 13 queens, the maximum number of safe squares is 47. Out of 9 176 358 300 744 339 432 possible combinations, this is achieved in 8.



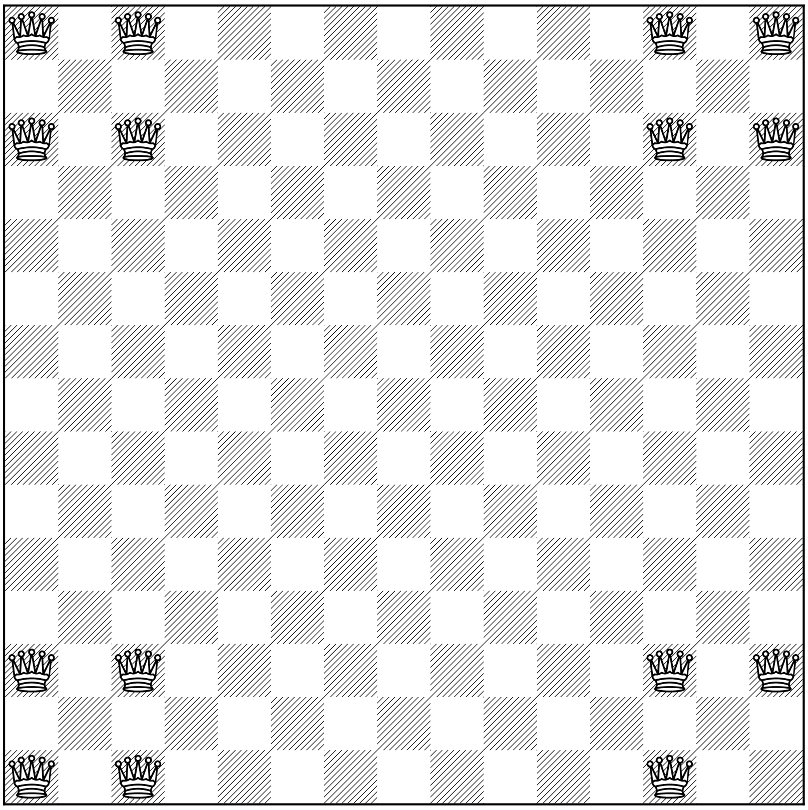
Size 14

On a board of size 14×14 with 14 queens, the maximum number of safe squares is 56. Out of 880 530 516 383 349 192 480 possible combinations, this is achieved in 20.



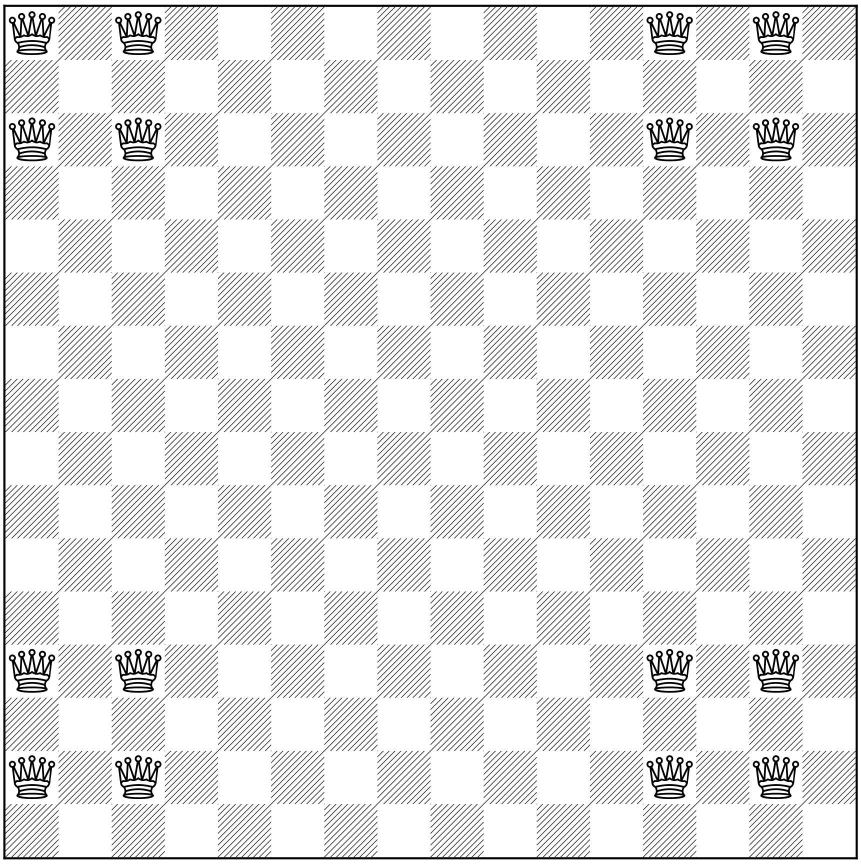
Size 15

On a board of size 15×15 with 15 queens, the maximum number of safe squares is 72. Out of 91 005 567 811 177 478 095 440 possible combinations, this is achieved in 16.



Size 16

On a board of size 16×16 with 16 queens, the maximum number of safe squares is 82. Out of 10 078 751 602 022 313 874 633 200 possible combinations, this is achieved in 4.



Larger sizes

For larger sizes, the number of safe squares has been calculated computationally but its maximality has not yet been proven.
The appendix has a table with all known numerical results.

History

This variant of the Anti-N-Queens problem has been studied before and the following is a non-comprehensive list of publications on the subject, in chronological order, covering the earliest known instances and the most recent ones, with newer results:

– Walter W. Rouse Ball, *Mathematical Recreations and Problems of Past and Present Times*, 3rd ed., MacMillin, 1896.

– Wilhelm E. Ahrens, *Mathematische Unterhaltungen und Spiele*, Teubner, Leipzig, 1901.

– Italo Ghersi, *Mathematica dilettevole curiosa*, Hoepli, Milan, 1913.

– Henry E. Dudeney, *Amusements in Mathematics*, Nelson and Sons, 1917.

– Stephen Ainley, *Mathematical Puzzles*, G. Bells and Sons, 1977.

– Mario Velucchi, in *Jouer Jeux Mathematiques*, no 19, 1995.

– Bernand Lemaire and Pavel Vitushjinkiy, *Placing n non dominating queens on the n X n chessboard*, Fédération Française des Jeux Mathématiques, 2011.

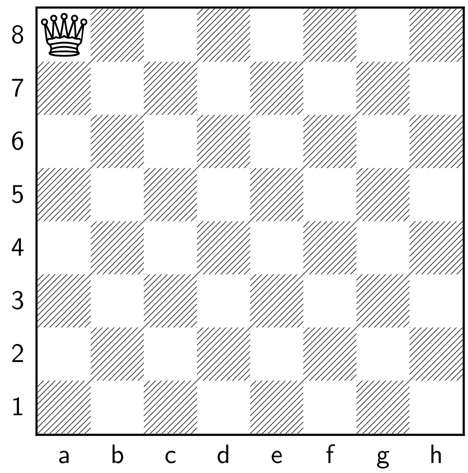
The series has been included in the Online Encyclopedia of Integer Sequences (OEIS) with the sequence identifier A001366.

Variant number of queens on 8×8 chessboard

We now explore another variation of the ANTI-N-QUEENS problem, namely having a standard chessboard of size 8×8 and varying the number of queens placed on it.
The possible combinations are $\binom{64}{N}$.

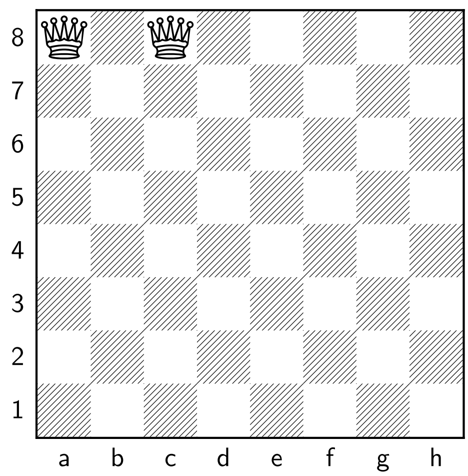
One queen

On a regular chessboard with 1 queen, the maximum number of safe squares is 42. Out of 64 possible combinations, this is achieved in 28.



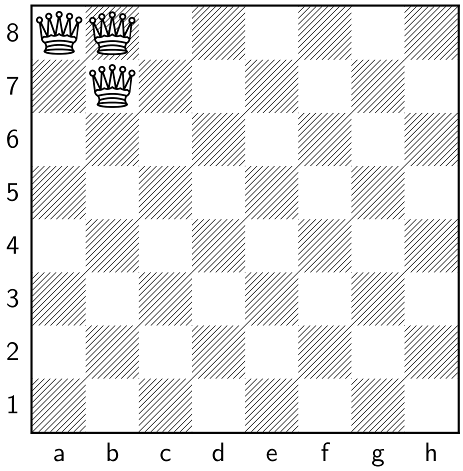
Two queens

On a regular chessboard with 2 queens, the maximum number of safe squares is 31. Out of 2 016 possible combinations, this is achieved in 48.



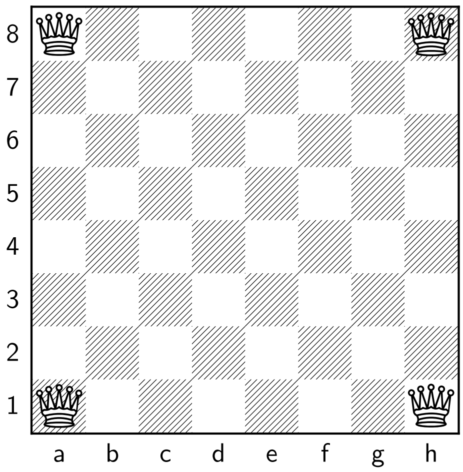
Three queens

On a regular chessboard with 3 queens, the maximum number of safe squares is 25. Out of 41 664 possible combinations, this is achieved in 24.



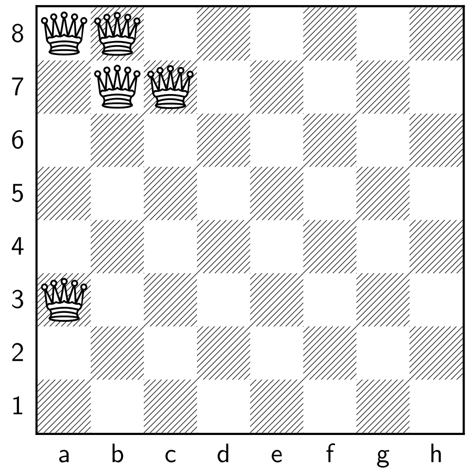
Four queens

On a regular chessboard with 4 queens, the maximum number of safe squares is 24. Out of 635 376 possible combinations, this is achieved in 1.



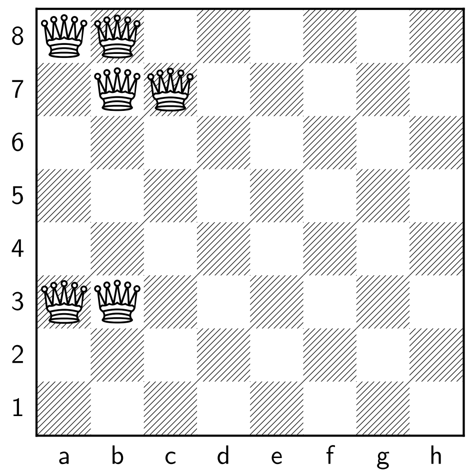
Five queens

On a regular chessboard with 5 queens, the maximum number of safe squares is 17. Out of 7 624 512 possible combinations, this is achieved in 16.



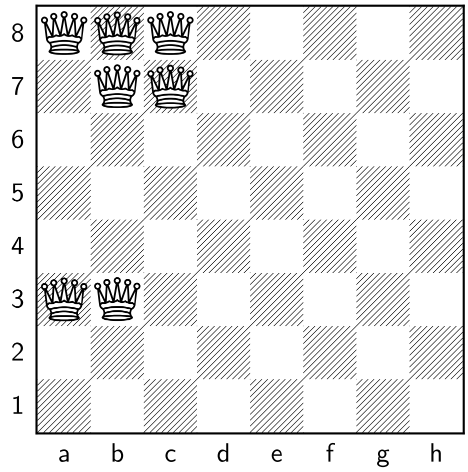
Six queens

On a regular chessboard with 6 queens, the maximum number of safe squares is 15. Out of 74 974 368 possible combinations, this is achieved in 16.



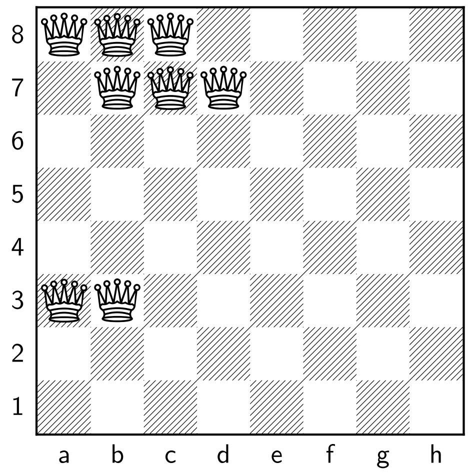
Seven queens

On a regular chessboard with 7 queens, the maximum number of safe squares is 13. Out of 621 216 192 possible combinations, this is achieved in 8.



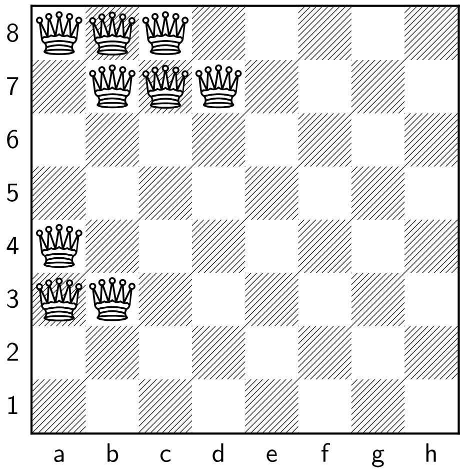
Eight queens

On a regular chessboard with 8 queens, the maximum number of safe squares is 11. Out of 4 426 165 368 possible combinations, this is achieved in 48.



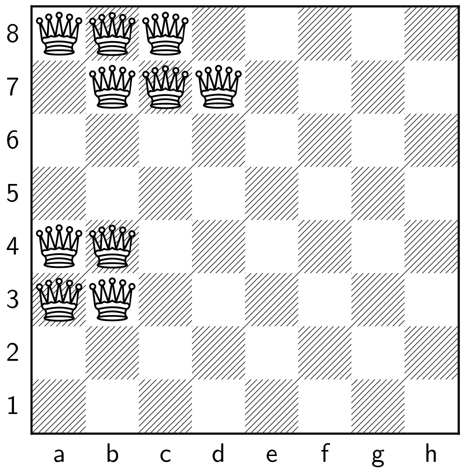
Nine queens

On a regular chessboard with 9 queens, the maximum number of safe squares is 10. Out of 27 540 584 512 possible combinations, this is achieved in 24.



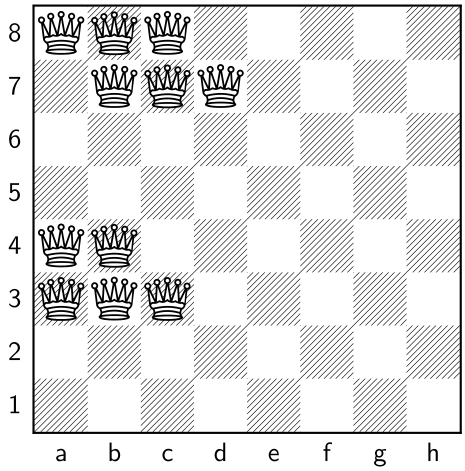
Ten queens

On a regular chessboard with 10 queens, the maximum number of safe squares is 9. Out of 151 473 214 816 possible combinations, this is achieved in 24.



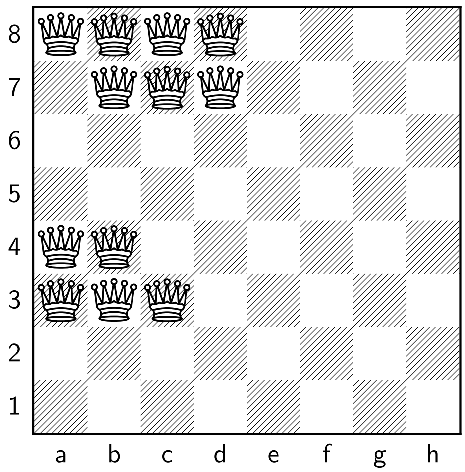
Eleven queens

On a regular chessboard with 11 queens, the maximum number of safe squares is 8. Out of 743 595 781 824 possible combinations, this is achieved in 48.



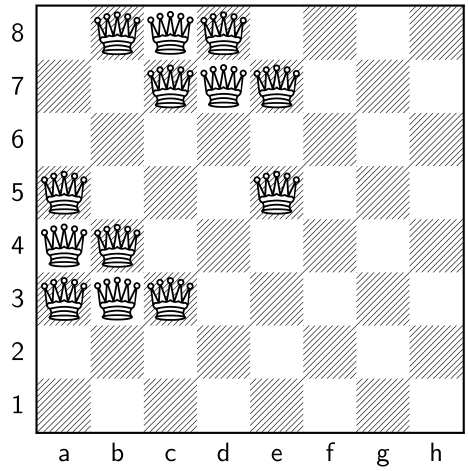
Twelve queens

On a regular chessboard with 12 queens, the maximum number of safe squares is 7. Out of 3 284 214 703 056 possible combinations, this is achieved in 256.



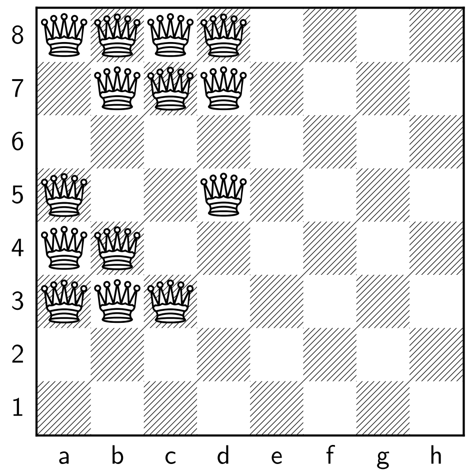
Thirteen queens

On a regular chessboard with 13 queens, the maximum number of safe squares is 7. Out of 13 136 858 812 224 possible combinations, this is achieved in 8.



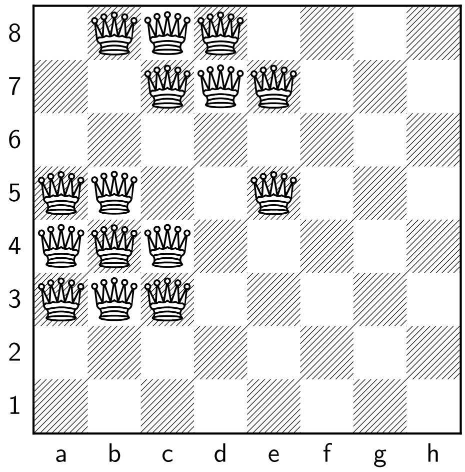
Fourteen queens

On a regular chessboard with 14 queens, the maximum number of safe squares is 6. Out of 47 855 699 958 816 possible combinations, this is achieved in 336.



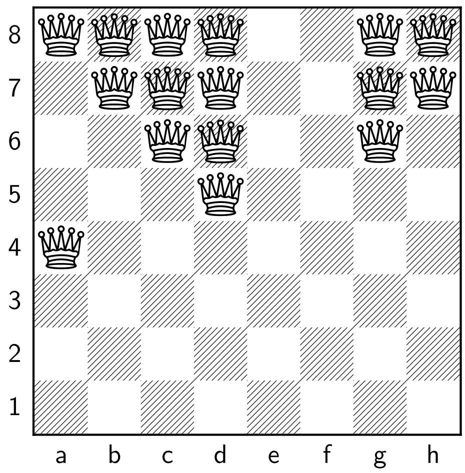
Fifteen queens

On a regular chessboard with 15 queens, the maximum number of safe squares is 6. Out of 159 518 999 862 720 possible combinations, this is achieved in 16.



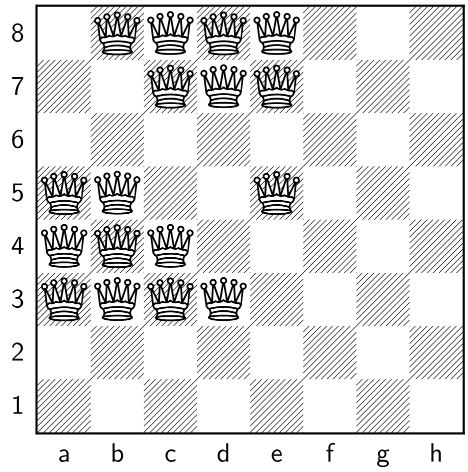
Sixteen queens

On a regular chessboard with 16 queens, the maximum number of safe squares is 5. Out of 488 526 937 079 580 possible combinations, this is achieved in 484.



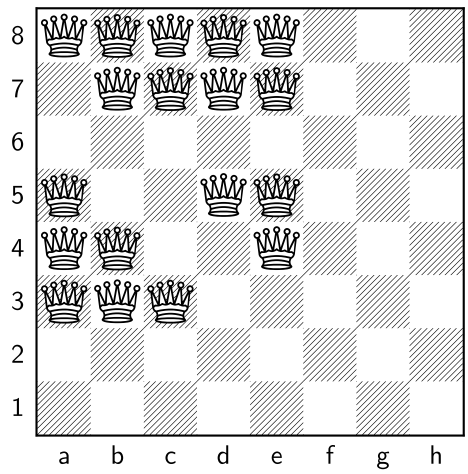
Seventeen queens

On a regular chessboard with 17 queens, the maximum number of safe squares is 5. Out of 1 379 370 175 283 520 possible combinations, this is achieved in 16.



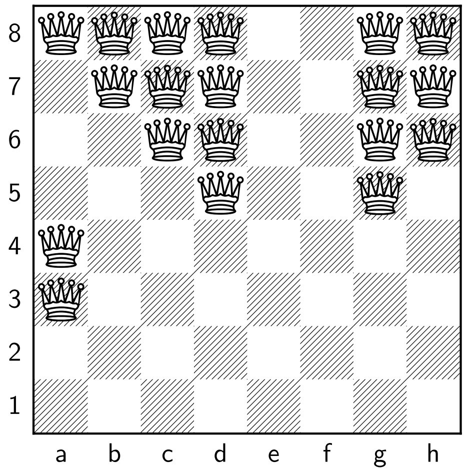
Eighteen queens

On a regular chessboard with 18 queens, the maximum number of safe squares is 4. Out of 3 601 688 791 018 080 possible combinations, this is achieved in 224916.



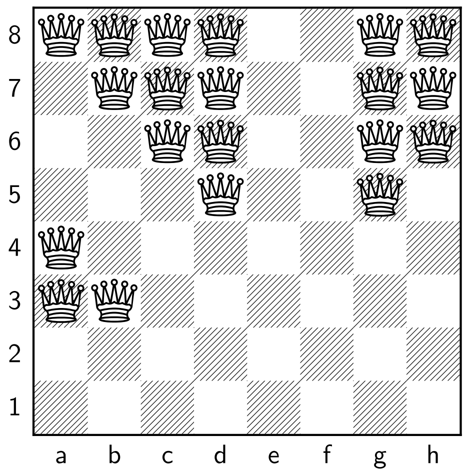
Nineteen queens

On a regular chessboard with 19 queens, the maximum number of safe squares is 4. Out of 8 719 878 125 622 720 possible combinations, this is achieved in 59104.



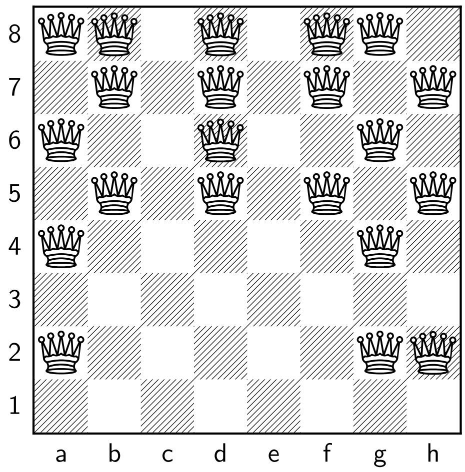
Twenty queens

On a regular chessboard with 20 queens, the maximum number of safe squares is 4. Out of 19 619 725 782 651 120 possible combinations, this is achieved in 12850.



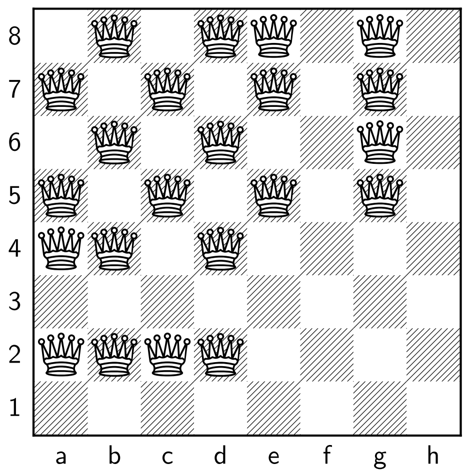
Twenty-One queens

On a regular chessboard with 21 queens, the maximum number of safe squares is 4. Out of 41 107 996 877 935 680 possible combinations, this is achieved in 2216.



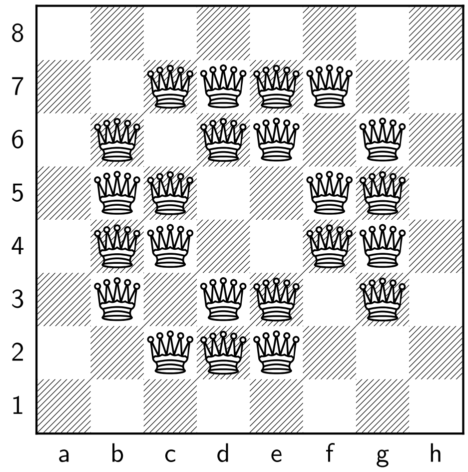
Twenty-Two queens

On a regular chessboard with 22 queens, the maximum number of safe squares is 4. Out of 80 347 448 443 237 920 possible combinations, this is achieved in 284.



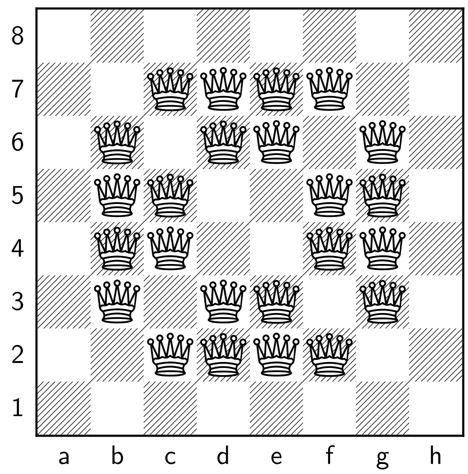
Twenty-Three queens

On a regular chessboard with 23 queens, the maximum number of safe squares is 4. Out of 146 721 427 591 999 680 possible combinations, this is achieved in 24.



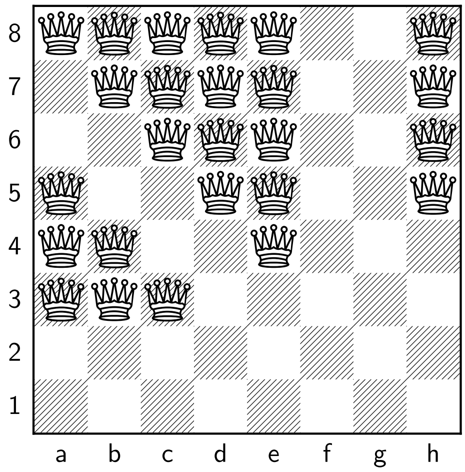
Twenty-Four queens

On a regular chessboard with 24 queens, the maximum number of safe squares is 4. Out of 250 649 105 469 666 120 possible combinations, this is achieved in 1.



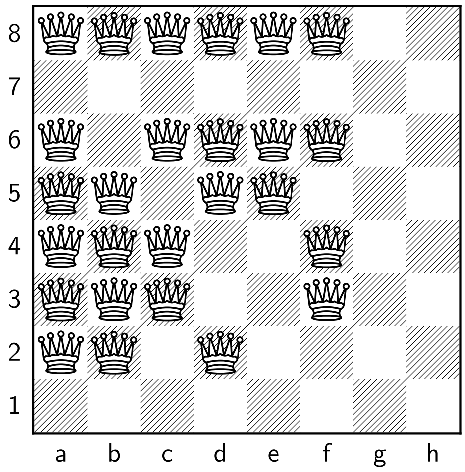
Twenty-Five queens

On a regular chessboard with 25 queens, the maximum number of safe squares is 3. Out of 401 038 568 751 465 792 possible combinations, this is achieved in 24.



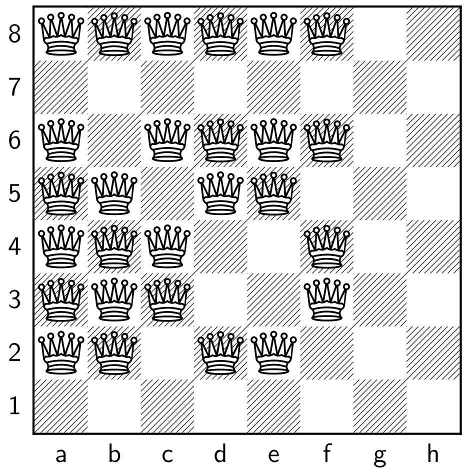
Twenty-Six queens

On a regular chessboard with 26 queens, the maximum number of safe squares is 2. Out of 601 557 853 127 198 688 possible combinations, this is achieved in 11768860.



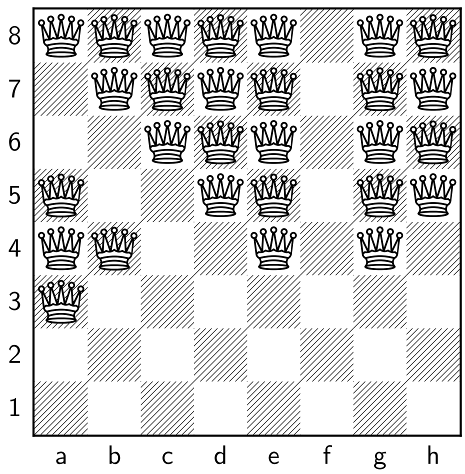
Twenty-Seven queens

On a regular chessboard with 27 queens, the maximum number of safe squares is 2. Out of 846 636 978 475 316 672 possible combinations, this is achieved in 2037336.



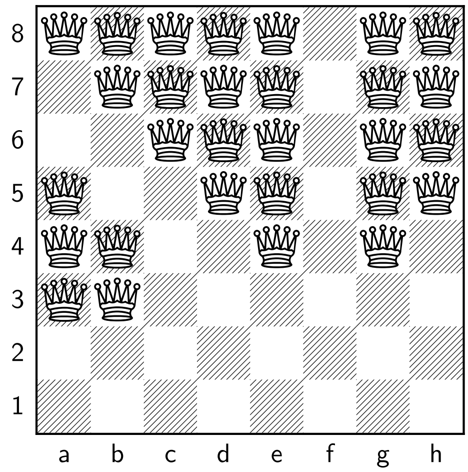
Twenty-Eight queens

On a regular chessboard with 28 queens, the maximum number of safe squares is 2. Out of 1 118 770 292 985 239 888 possible combinations, this is achieved in 271522.



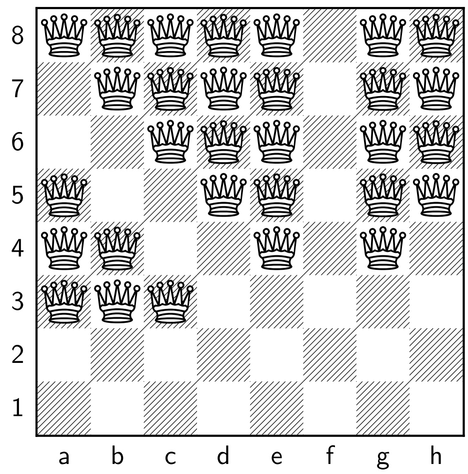
Twenty-Nine queens

On a regular chessboard with 29 queens, the maximum number of safe squares is 2. Out of 1 388 818 294 740 297 792 possible combinations, this is achieved in 26128.



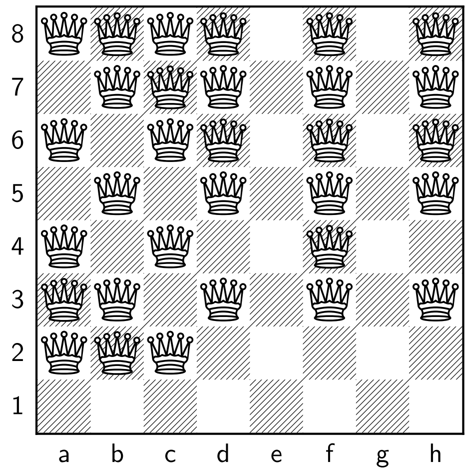
Thirty queens

On a regular chessboard with 30 queens, the maximum number of safe squares is 2. Out of 1 620 288 010 530 347 424 possible combinations, this is achieved in 1614.



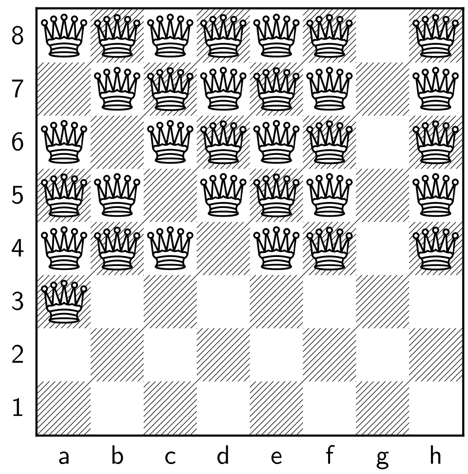
Thirty-One queens

On a regular chessboard with 31 queens, the maximum number of safe squares is 2. Out of 1 777 090 076 065 542 336 possible combinations, this is achieved in 48.



Thirty-Two queens

On a regular chessboard with 32 queens, the maximum number of safe squares is 1. Out of 1 832 624 140 942 590 534 possible combinations, this is achieved in a very large number.



Larger number of queens

The cases with a larger number of queens on the board can be deduced from the data already presented.
Based on the *Observation 3* that queens and safe squares can be exchanged, we can easily see that any number of queens up to 42 will result in the maximum of one safe square. This is due to the fact that placing a single queen can result to a maximum of 42 safe squares.
An even larger number of queens (up to 64) will result in no safe squares on the board.
The number of possible combinations is also known, since $\binom{M}{N} = \binom{M}{M-N}$. Therefore, the number of different placements of 33 queens, for example, will be the same as the number of different placements of 31 queens $(64 - 33 = 31)$. This can be intuitively understood as finding all the different placements of squares *without* a queen.
The appendix has a table with the complete numerical results.

Conclusion

This paper presented results on the ANTI-N-QUEENS problem.
The first part included all known results for N queens on a $N \times N$ board.
The second part included all results for a variant number of queens placed on a standard 8×8 chessboard.
The series of numbers in this second result, being an novel result, has been submitted and is now included in the Online Encyclopedia of Integer Sequences (OEIS) with the sequence identifier A342151.

Bibliography

Relevant sources have been listed in the subsection named “History” above.

Appendix A: Results for boards of variant size

N	safe	occ.
1	0	–
2	0	–
3	0	–
4	1	184
5	3	8
6	5	24
7	7	304
8	11	48
9	18	4
10	22	8
11	30	16
12	36	42
13	47	8
14	56	20
15	72	16
16	82	4

N	safe(est.)
17	97
18	111
19	132
20	145
21	170
22	186
23	216
24	240
25	260
26	290
27	324
28	360
29	381
30	420
31	442
32	485

N	safe(est.)
33	530
34	554
35	603
36	650
37	702
38	731
39	785
40	841
41	873
42	932
43	993
44	1056
45	1091

Appendix B: Results for N queens on 8×8 board

N	safe	occ.
1	42	28
2	31	48
3	25	24
4	24	1
5	17	16
6	15	164
7	13	8
8	11	48
9	10	24
10	9	24
11	8	48
12	7	256
13	7	8
14	6	336
15	6	16
16	5	484
17	5	16
18	4	224916
19	4	59104
20	4	12850
21	4	2216
22	4	284
23	4	24
24	4	1
25	3	24
26	2	11768860
27	2	2037336
28	2	271522
29	2	26128
30	2	1614
31	2	48
32–42	1	
43–64	0	